THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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ON VON STAUDT'S GEOMETRIE DER LAGE.

(*Continued from page* 331.)

THE extent of the complete domain can be estimated by comparing it with its real part, that is, with the visible universe. For the sake of this comparison, von Staudt speaks as though it were possible to specify the number of real elements in any base, but gives an explicit warning as to the meaning he attaches to the symbol n which he uses for an infinitely great quantity. The number of real points on a line he denotes by $n+1$; the *one* is apparently the ideal point, the n belonging to the actual universe. Now every imaginary pair can be assigned by means of a harmonic cast with one point, A, fixed. If this be $(ABA'B')$, we have a choice of $\frac{1}{2}n(n-1)$ pairs for BB', but no further choice; the number of imaginary points is therefore $2 \times \frac{1}{2}n(n-1)$, that is, $n(n-1)$. Hence if the number of real points on a line is denoted by $n+1$, the total number of points on the line is n^2+1. On an imaginary line of the first kind the points are one real, n imaginary; on an imaginary line of the second kind all are imaginary.

To obtain the number of points in a real plane, take one real point, O, arbitrarily; all the real points of the plane are on the $n+1$ real lines through O, and as on each such line there are n real points, exclusive of O, the number of real points in a real plane is seen to be $n(n+1)+1$, that is, n^2+n+1. This is also the number of the real lines; and as every real line yields $n(n-1)$ imaginary points, the number of these is $n(n-1)(n^2+n+1)$, that is, n^4-n, giving, for the total number of points in a plane, n^4+n^2+1. If the plane is imaginary, there are $n+1$ real points, n^4+n^2-n imaginary. Similarly the number of any other kind of element in any primitive form is expressed in terms of n; and von Staudt points out, that if the number of real elements in

any real base is $f(n)$, the number of real and imaginary together is $f(n^2)$.

His expressions for the numbers of real points in a real line, plane, space, are $n+1$, n^2+n+1, n^3+n^2+n+1. Now as there is one ideal point in every line, one ideal line and consequently $n+1$ ideal points in every plane, one ideal plane with its n^2+n+1 points in space, these give, for the numbers in the actual universe, n, n^2, n^3. Similarly the numbers of real and imaginary points together, in the complete line, plane, and space, are n^2, n^4, n^6, when the ideal elements are excluded.

The total number of lines in space, real and imaginary, is found to be $n^8+n^6+2n^4+n^2+1$, of which $n^4+n^3+2n^2+n+1$ are real, $n^7+n^6+n^5-n^3-n^2-n$ are imaginary of the first kind, $n^8-n^7-n^5+n^4$ imaginary of the second kind; excluding ideal lines, there are $n^8+n^6+n^4$ in all, of which $n^4+n^3+n^2$ are real. These may also be enumerated as a line p, $n^2(n^2+1)^2$ which cut p, n^8 which do not cut p (*B.* 86-91).

Thus the extent of the domain is determined; it still remains to form some conception of its nature. This requires us to consider the interconnection of the elements, and their arrangement. As to the interconnection of the elements, it is shown that all projective operations are possible; all harmonic properties exist; figures (curves and surfaces) are determined by combinations of elements exactly as in the restricted real domain; it may even be that imaginary elements yield a real result. Two primitive forms of the first grade are projectively correlated by means of three elements in each, and there is no distinction between real and imaginary in the selected three (*B.* 91-119).

When a conic and a straight line in its plane do not visibly intersect, they determine a pair of imaginary points as double elements of the involution of conjugate points on the line; and when two conics lie in one plane they determine four points, all real, two real with one pair of conjugate imaginaries, or two pairs of conjugate imaginaries; similarly as regards the common tangents (*B.* 119-126). Thus no different class of results is obtained in the enlarged domain; the dependence of the elements on one another is of the same nature as the dependence of real elements on one another.

More light is thrown on the arrangement of the elements by a minute study of the nature of some one complete primitive form. To obtain any such conception of the complete field as we have of its real part, we consider the possibility of representing every element by a real element in some real space. As precisely one real line passes through every imaginary point, it is conceivable that the imaginary elements belonging to any form can be made to yield a series of real elements. In considering such a representation, von Staudt deals in the first instance with the

imaginary line of the second kind; in the discussion he relies mainly on the properties of the regulus, this supplying the most convenient means of comparing projective ranges whose bases are not coplanar. Two such ranges, $PQRS\ldots$, $P'Q'R'S'\ldots$, on lines u, v, determine a regulus $pqrs\ldots$; and the rays of this are all met by another system of straight lines $wxy\ldots$, forming the associated regulus. The two systems exist on one ruled surface, and every ray of one system intersects every ray of the other. The regulus can equally well be determined as the system of lines all meeting three given lines uvw; or as the system of lines given by two projective axial pencils whose bases are not concurrent.

Regarding it as determined in this last manner, corresponding planes giving the lines $pqr\ldots$, let u, x, be any two lines meeting p, q, r, that is, any two generators of the second system; the ranges determined on u, x, are in perspective with each axial pencil, and are thus seen to be projective. Hence to project u into x by a direct projection, we determine a certain axial pencil whose planes pass through the points P, Q, R, of u, and cut this by the line x; these two steps are combined in one by means of the regulus. In using the regulus to pass from one straight line to another, we are employing only the ordinary processes of projection.

An arbitrary plane cuts the regulus in a conic; and this is projectively connected with the range $PQRS\ldots$ on u by means of a suitable axial pencil whose planes pass through the rays $pqrs\ldots$, which determines in the arbitrary plane a flat pencil whose vertex is on the conic. Thus the line is projectively compared with the conic by the usual processes of linear projection; while it is projected into the conic by a skew projection. If however the cutting plane contains a generator t of one regulus, the remainder of the section is a generator x of the associated regulus.

The imaginary straight line of the second kind, u, has a conjugate u', which does not intersect it; through every point P on u one real line passes, which joins P to its conjugate P' on u'. Thus through the various points $PQRS\ldots$ of u there are real lines $pqrs\ldots$, no two of which can intersect, as if they did, they would determine a plane in which both u and u' would lie. Through any real point in space, X, one of these lines must pass, given by the two conjugate planes Xu, Xu'. Cutting this system of real lines by an arbitrary real plane, the points of the imaginary line u are represented singly by the real points of this real plane, with the exception only of the points on the line U_1U_1', where U_1, U_1', are the points in which u, u' meet the plane.

Consider all the real points $P_1Q_1R_1\ldots$ on any real line x in this plane; the lines of the system that pass through $P_1Q_1R_1\ldots$ are

all lines that meet three given lines u, u', x; they are therefore rays of the regulus determined by any three of them, p, q, r. Hence points $PQRST\ldots$, on the imaginary line u can be represented by points on a real straight line in this way—that is, can be projected into real points on a real straight line—only if the real lines through them, $pqrst\ldots$, belong to one regulus. Points having this property, that is, all points on a straight line of the second kind whose real bases belong to one regulus, form what von Staudt calls a *Kette* (chain or strand); a set of four points of this is spoken of as a neutral cast. Four points on an imaginary straight line of the second kind form a neutral cast if the four real lines through them are generators of a regulus; or, we may conveniently say, if these four real lines form a neutral set (*B.* 137-142). As three non-intersecting lines determine a regulus, a strand is determined by three of its points, which agrees with the fact that any three collinear points, real or imaginary, can be projected into any three collinear points.

In order not to omit any part of u in forming the strands, but to avoid unnecessary repetition, we keep P, Q fixed, and let R vary. We thus obtain a regulus for any position of R, and this is cut by the arbitrary real plane in a conic. As the regulus contains p, q, this conic passes through P_1, Q_1, the points in which the plane cuts p, q; and as every point on u or u' lies on a ray of the regulus, the conic passes through the points U_1, U_1', in which the plane cuts u, u'. Thus the different strands that pass through P, Q, are represented by the conics through P_1, Q_1, U_1, U_1', and of these one, namely, the one determined by any point on P_1, Q_1, is the line-pair P_1Q_1, U_1U_1'. That is to say, one strand through PQ is projected into the real straight line P_1Q_1 by the ordinary processes of projection, and all other strands through PQ are projected into conics by a skew projection, the conics passing through the two real points P_1Q_1 and the conjugate imaginary points U_1U_1'. Similarly, choosing a different pair PQ, we have a different set of strands.

The strand that is projected into a straight line can be made to be any one, by a proper choice of the arbitrary plane. For let t be any real ray of the regulus $pqr\ldots$ which determines the strand; then any real plane through t cuts the regulus in a generator x of the associated system, which is the real representation of the strand.

Thus a strand has precisely the same projective properties as the real part of a real line: and the complete form, the imaginary straight line of the second kind, is made up of interlaced strands, no two of which can have more than two points in common. By any one strand the whole figure is given, all the other strands can be inferred; thus the one given strand can be treated as the figure. The strands through any two points form a singly in-

finite system; they can be represented by conics through four fixed points, that is, by a pencil of conics. The relation of the strands to one another, and to all the points of the complete form, is exactly represented by the relation of these conics to one another and to the real points of the plane.

Now all the points of a real line, or of an imaginary straight line of the first kind, can be projected into the points of an imaginary straight line of the second kind; hence the results just obtained apply to any straight line. Any straight line, regarded as a field of points, is thus seen to be highly complex, consisting of a triply infinite system of strands. If we wish to avoid using the points of the line more than once, while including all, we may say that the line AB is composed of a singly infinite system of strands through AB. Thus it is doubly true that a definite segment of a line is specified by three points ABC; the third point C specifies a particular strand through AB, and also a particular one of the two segments into which this strand is divided by A, B (*B.* 138).

The points of a real line u can be represented on a real plane through the line by a very simple device. Let I be an imaginary point in this plane, on a real line distinct from u, meeting u in Z; joining I to any point P of u we have an imaginary line of the first kind, on which there is necessarily one real point. If P is real, this one real point is none other than P itself; if P is imaginary, this real point, S, is not on u. In this way every point of the line u is represented by a real point of the real plane through u, and every real point of the plane represents a point of the line, real or imaginary. The only exception to this statement is offered by the points on the real line through I, all of which represent Z.

For the actual construction of the points S, S', which represent the imaginary points P, P' given by the real cast $(ABA'B')$, we use also I', conjugate to I; the lines $I'P'$, $I'P$, being conjugate to IP, IP', meet them in S, S'. Now $I\,(PP', AA')$ and $I'\,(PP', AA')$ are harmonic pencils; hence

$$I\,(P'P, AA') = I\,(PP', AA') = I'\,(PP', AA');$$

that is, as IP' and $I'P$ meet in S', and IP, $I'P'$ in S, the six points $II'AA'SS'$ are on a conic, and in this conic SS', AA' are conjugate chords. Similarly, SS', BB' are conjugate chords in a conic $II'BB'SS'$.

Though von Staudt does not give the complete projective theory of metric relations, he nevertheless makes some use of the circular points in his metric supplements, and they simplify this construction in practice. Considering the two points I, I' as the circular points, the two conics become circles; SS' is their common chord, hence its poles lie on the line of centres, which is therefore

the line u on which AA', BB' lie. The two circles are consequently on AA', BB' as diameters, which gives the usual construction for the two points that represent the double points in the case of an elliptic involution (*B.* 264-5).

In this representation of the imaginary points of a line by means of the real points of a plane, the point S_1 is on one side or the other of the real strand $P_1Q_1R_1 \ldots$; and if S_1, S_1' are a conjugate pair given by the harmonic cast $(P_1Q_1P_1', Q_1')$, then S_1S_1' lie on opposite sides of P_1Q_1. Similarly two imaginary points may be separated by any other strand. Thus an element S can belong to the strand PQR, and is then said to be neutral towards PQR; or it can lie on one side or the other. We can compare the positions of two points, saying that they are on the same side or on opposite sides of the strand; that is, having distinguished the sides by means of one pair X_1X_2, we can say that S_1 is on the same side as X_1, or on the same side as X_2.

Thus at times von Staudt regards two conjugate imaginary points as lying on opposite sides of the real line to which they belong, that is, on opposite sides of a strand. But the conception of one side or the other is essentially graphic. To make the distinction universally applicable, this conception must be replaced by one that is purely intellectual, and this is supplied by the sense of a form. On the base of any real primitive form of the first grade the distinction of sense is, roughly, the distinction between this way and that way; if A, B, C are, for example, three points of a real range, the whole range can be described in the sense ABC or in the sense CBA. There is no general way of comparing sense, absolutely, in two forms on two different bases; if however the elements of the primitive form on the second base determine on the first base a form of the same nature as the given one on this base, then these two primitive forms on the first base, taken in the same sense, do determine what is to be understood by the same sense in the two different bases. This direct comparison is possible for a range and a flat pencil, in a plane; an axial pencil and a flat pencil, in a sheaf; a range and an axial pencil, in space; provided in every case that the base of the one form is not an element of the other form (*B.* 30).

In all other cases, the comparison of sense must be with reference to some external element (*B.* 31). For example, two coplanar ranges can be compared with reference to a point in their plane by means of the flat pencils that they determine at this point; the ranges can be said to be described in the same sense or in opposite senses with reference to this point. Similarly two coplanar flat pencils can be compared with reference to a line in their plane, by means of the ranges they determine on the line; two concentric flat pencils (in different planes), with reference to

a line through their common centre, by means of axial pencils; two axial pencils whose bases meet in a point, with reference to a plane through this point, by means of two flat pencils.

Thus the comparison of sense in two different forms in most cases depends on another element; and we can divide all elements of one kind in a given field into two classes, by means of two suitably given forms in that field. Selecting arbitrarily a sense in each form, and calling these the same sense, we have in the one class all elements with reference to which the two forms are described in the same sense, and in the other, all elements with reference to which the two forms are described in opposite senses. In this way all the points in a plane are divided by two lines in the plane, regarded as bases of ranges, into two classes; when the lines are real, these two classes contain the points lying in one pair of angles or in the other. All the lines in a plane are divided by two points, regarded as bases of two flat pencils in the plane, into two classes; namely, as regards real elements, those meeting the line joining the points in one segment or the other (*B.* 32).

Sense on a regulus can be compared with sense on a line, unless the line intersects two distinct rays of the regulus in real points. This exception, however, does not affect the use that von Staudt makes of this in discriminating between imaginaries. Regarding the line as a range, its sense can be compared with that of any axial pencil perspective with the regulus; and regarding the line as an axial pencil, it can be compared as to sense with any range perspective with the regulus (*B.* 34, 39).

The associated regulus $xyz\ldots$, whose rays are simply all lines that meet $pqr\ldots$, can also be compared with the line as to sense. Von Staudt shows that if with reference to a line s, regarded as the base of a range, $pqr\ldots$ and $xyz\ldots$ are described in the same sense, then with reference to the line regarded as the base of an axial pencil, they are described in opposite senses (*B.* 40). All lines in space that do not cut the ruled surface (determined by $pqr\ldots$ or $xyz\ldots$) in two distinct points fall into two classes; in the one are those lines with reference to which, regarded as the bases of ranges, the regulus $pqr\ldots$ and the regulus $xyz\ldots$ are described in the same sense, in the other are the lines with reference to which $pqr\ldots$ and $xyz\ldots$ are described in opposite senses. If the lines are regarded as bases of axial pencils, the words same and opposite are interchanged, but the partition of the lines is unaffected (*B.* 35-37, 40). Lines belonging to different classes are said to be separated by the ruled surface (or by the regulus); lines belonging to the same class are not separated. The distinction arising from taking the sense into account is exactly the distinction, in the case of real lines, between one side and the other of the ruled surface (*B.* 43); if two lines are

separated by the surface, and the two systems $pqr\ldots$, $xyz\ldots$, are described in the same sense with reference to one line, they are described in opposite senses with reference to the other line.

Returning to the real representation of the imaginary line of the second kind by means of real lines through the points $PQRS\ldots$, it is seen that two points S, S', determined by the harmonic cast $(PQP'Q')$, lie on opposite sides of the strand PQP'; the lines s, s' are separated by the surface (regulus) pqr. Keeping the sense of the associated regulus $xyz\ldots$ unchanged, the two lines s, s' are discriminated by the sense in which $pqr\ldots$ (or $PQR\ldots$) is described with reference to them; one, s, lies in the sense pqr, the other, s', in the sense rqp; and as to the points S, S', these lie in the sense PQR, RQP respectively. If however s belongs to the regulus $pqr\ldots$, then S is said to be neutral with respect to PQR (*B.* 131, 2).

This is the *a posteriori* justification for discriminating between two conjugate imaginary elements by associating with the form the idea of sense; it depends on the representation of the imaginary elements by means of real elements, and is seen to have a real geometrical foundation.

Thus von Staudt, having adjoined to the visible universe a more extensive region, proves that the enlarged domain is a coherent and manageable whole; that there is no essential difference between the elements recognised by the bodily senses and those apprehended by pure intellect. The fact that there is not a single diagram in his two books throws an interesting side-light on his conviction of this identity of nature. Having freed himself completely from the trammels of reality, he proceeds to the undisturbed consideration of the geometry of curves and surfaces of the second order, and of cubic and quartic curves in space, but it hardly seems necessary to enlarge upon this here, as his treatment does not differ materially from that in more recent works on geometry. The matters that are of special interest in von Staudt are his definition and explanation of his geometrical domain, which I have tried to set forth in these few pages, and his association of number with the elements of this domain, so as to arrive at a geometrical definition of number; upon this I have not attempted to enter.

C. A. Scott.

MATHEMATICAL NOTES.

80. [L². 21. d.] *On the Cylindroid.*

The equation of this surface is $z(x^2+y^2)-2mxy=0$.

Consequently the planes $y=x\tan\phi$ and $z=m\sin 2\phi$ intersect in a straight line every point of which lies on the surface; for eliminating ϕ we have the equation to the surface.

Hence the equation to a generator is

$$\frac{x-o}{1}=\frac{y-o}{\tan\phi}=\frac{z-m\sin 2\phi}{o}, \quad \text{.....(A)}$$

where ϕ is the variable parameter.

Every generator therefore intersects the axis of z and is parallel to the plane of xy : also the whole surface lies between the planes $z=\pm m$.

The equation of a plane through α, β, γ perpendicular to a generator is

$$(x-\alpha)+(y-\beta)\tan\phi=0, \quad \text{.....(B)}$$

The equations (A) and (B) combined give the coordinates of the feet of the perpendicular from α, β, γ upon the generator.

Hence $\quad (x-\alpha)+(y-\beta)y/x=0\,; \quad x^2+y^2-\alpha x-\beta y=0,$

a right circular cylinder.

Also $\left.\begin{aligned} x&=(\alpha+\beta\tan\phi) \quad /(1+\tan^2\phi)\\ y&=\tan\phi(\alpha+\beta\tan\phi)/ \ldots\ldots\\ z&=2m\tan\phi \quad / \ldots\ldots \end{aligned}\right\}$, whence $\beta x+\alpha y-(\alpha^2+\beta^2)z/2m=\alpha\beta.$

Thus the feet of the perpendiculars all lie on the section of a right circular cylinder by a plane, *i.e.* a plane ellipse. R. F. Davis.

81. [R. 4. d.] *On Note* 75, p. 319.

These quadrilaterals are discussed by Mr. Dobbs in his *Elementary Geometrical Statics*, pp. 185-193, where will also be found a series of interesting properties connected therewith.

82. [L¹. 10. d.] *Elementary Note on the Parabola.*

Let P be a point on the parabola, focus S, and X be the foot of the directrix; and take upon the axis $NR=SN$. Then, assuming the usual definition, a circle described from P as centre and SP as radius will pass through S and R and touch the directrix KX in M. Hence (Euclid III. 36)

$$MX^2=XS\,.\,XR \quad \text{or} \quad PN^2=4AS\,.\,AN. \qquad \text{R. F. Davis.}$$

83. [L¹. 7. d.] *On Note* 79, p. 337.

The equations $a^2\cot A+\beta\gamma\operatorname{cosec}A=$ etc. represent rectangular hyperbolas, concentric with the given ellipse, the centre being the symmedian point of the triangle. Each curve passes through an angle of the triangle (one through each), and touches the median line at that point; and as the centre is on the symmedian line, its asymptotes are parallel to the internal and external bisectors of that angle. Curves having these properties can be shown by geometry to pass through the foci of the given ellipse. E. P. Rouse.

PROBLEMS.

370. [M¹. 6. h. α.] (i) A circle is described on a variable chord PQ through the cusp of a cardioide. Find the locus of the centres of similitude of this circle and the director circle of the cardioide.

[M¹. 5. c. β.] (ii) In a right cissoid, find a relation between the area of the generating circle and the areas of the first and second pedals with respect to a cusp. E. N. Barisien.

371. [L¹. 3. a.] Given two points on an ellipse and the positions of a pair of conjugate diameters, find the positions of the axes. A. Lodge.

372. [K. 13. a.] A plane figure $ABC\ldots$ receives any rigid-body displacement in space to the position $A'B'C'\ldots$. Prove that the mid-points of AA',

BB', CC', ... are co-planar, and that the planes bisecting these lines at right angles have a common point. C. E. M'VICKER.

373. [I. 1.] (i) Assuming that -1 has two square roots, prove (if possible) that it has no other square roots. (ii) Assuming that -1 has two and only two square roots, prove that the same is true of any number, real or complex. F. S. MACAULAY.

374. [L¹. 3. a.] Find the equation to the axis of a parabola touching an ellipse at the extremities of a normal chord. E. M. RADFORD.

375. [K. 20. e.] A famous stiltwalker bets that he will walk into the sea at low tide on a pair of stilts and remain there with his feet unwetted until high tide. Learning that he has miscalculated the height of the tide, he consults a mathematical friend, who measures a base of 50 feet on the sand, and then observes the angles, which the lowest point to which the tide falls on a perpendicular inaccessible rock makes with the ends of this base, to be $63° 4'$ and $58° 2'$ respectively. At the same time he observes the angles made with the same points by the highest perpendicular point at which the seaweed grows to be $63° 20'$ and $58° 16'$ respectively, and also the angle subtended at the first end of the base by the two points of rock to be $23° 18'$. Find the height of the tide. T. ROACH.

376. [A. 3. b.] (i) Solve completely the sextic whose roots are the ratios of the roots of the cubic x^3+px-q, one root of the sextic being given.

[A. 3. c.] (ii) Find the sum of the p^{th} powers of those roots of $x^{2q+1}+1=0$, the real parts of which are positive, (i) q even, (ii) q odd. A. S. TOLLER.

377. [K. 10. b.] (A), (B) are two circles, radii a, b and central distance d. PQ is a diameter of (A). Show that the polars of P, Q, with respect to (B), intersect on the polar of A. Discuss the conical envelope of the above polars, and the circular locus of the points of intersection of PB, QB with the respective polars. R. TUCKER.

378. [J. 1. b.] A triangle lies wholly within a quadrilateral. How many straight lines can be drawn to join an angular point of the quadrilateral to an angular point of the triangle, so that no two shall intersect each other or a side of the triangle, except at an angular point? St. John's (C.), 1891.

379. [R. 4. d.] Two circular rough-rimmed discs touch in a vertical plane, with an endless string tied tight round their rims, the lower disc touching a smooth horizontal plane. If A, B be the centres of the upper and lower discs, and the C.G. of the upper be at A, and that of the lower in AB produced, draw a diagram to show the positions and intersections of the forces on each disc—the action of the string on each disc being counted as a single force. Find also the ratios of the tensions in the two straight portions of the string. St. John's (C.), 1891.

SOLUTIONS.

UNSOLVED QUESTIONS.—57, 129, 144, 152, 171, 175, 252, 271, 275, 287, 306-7, 311, 320, 326-7, 336-8, 341, 344-7, 349, 356, 364, 369.

The question need not be re-written; the number should precede the solution. Figures should be very carefully drawn **to a small scale** on a separate sheet.

Solutions will be published as space is available.

293. [J. 2. f.] *If two random points are taken within a given triangle, show that their join is equally likely to cut any pair of sides of the triangle.*

C. SANDBERG.

Solution by C. E. M'VICKER.

Take an equilateral triangle $A'B'C'$ "corresponding" to ABC; A' to A, B' to B, C' to C, and any point P' in $A'B'C'$ to a point P in ABC, such that the areas

$$BPC : CPA : APB = B'P'C' : C'P'A' : A'P'B'.$$

Then either analytically, or by the theory of orthogonal projection, we see that when three points of one triangle lie in a line their "correspondents" also lie in a line. Hence if P', Q' in $A'B'C'$ correspond to P, Q in ABC, the line $P'Q'$ corresponds to PQ in the sense that to each point of it corresponds a point of PQ.

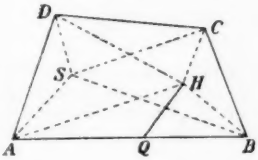
Now from the symmetry of the equilateral triangle the property is true for it. Therefore it is true for the other triangle.

By similar reasoning, if two random points are taken within a tetrahedron their join is equally likely to meet any pair of faces, and if three random points are taken within it, their plane is equally likely to meet any edge.

308. [L$_1$. 16. a.] *An ellipse is inscribed to a convex quadrilateral $ABCD$. If α denote the ratio of the product of the focal distances of A to the product of the sides meeting at A, with similar notation for the remaining vertices, then*

$$\alpha+\beta=\beta+\gamma=\gamma+\delta=1.$$

C. E. M'VICKER.



Solution by PROPOSER.

Draw HQ, making $\angle AHQ = ADS$. Then $\because \angle SAD = QAH$, we have by similar triangles

$$\alpha \equiv \frac{AS\,.\,AH}{AB\,.\,AD} = \frac{AQ\,.\,AD}{AB\,.\,AD} = \frac{AQ}{AB}$$

Again, $\angle BSC = \pi - ASD = BQH$,

and triangles BQH, BSC are therefore similar; so that

$$\beta \equiv \frac{BS\,.\,BH}{BA\,.\,BC} = \frac{BQ\,.\,BC}{BA\,.\,BC} = \frac{BQ}{BA}; \text{ hence } \alpha+\beta=1.$$

313. [K. 2. c.] *Through a given point in the plane of a triangle to draw three lines terminated by the sides of the triangle, two and two, such that their extremities shall lie on a circle.* J. A. THIRD.

Solution by C. E. YOUNGMAN.

It is known (see *Proc. London Math. Soc.*, xv. 122, xvi. 185) that for any point P in the plane of ABC there exist two triangles $A_1B_1C_1$ and $A_2B_2C_2$ (say T_1 and T_2) of fixed shapes, with their A's moving on AP, their B's on BP, and C's on CP, which have the property that the sides of either cut the non-corresponding sides of ABC in six concyclic points: further, that T_1 and T_2 give the same series of circles, one of which (the minimum) is the major auxiliary circle of the conic which touches BC, CA, AB on AP, BP, CP respectively.

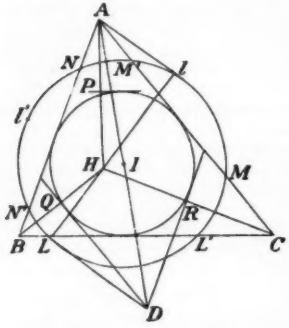
Now P being given, this conic is determined, and $\therefore$ its auxiliary circle, and $\therefore$ one position of T_1 and of T_2; and parallels through P to the sides of T_1 and T_2 give two solutions of the problem; for when T_1 or T_2 shrinks into P these parallels are its sides.

334. [K. 2. b. e.] *H is the orthocentre of a triangle ABC. Six tangents are drawn to the incircle perpendicular to BC, CA, AB and meeting them in the*

pairs of points L and L', M and M', N and N' respectively. Three tangents are drawn to the incircle perpendicular to AH, BH, CH and meeting these lines in P, Q, R respectively. The connectors LH, L'H meet the circle having AH as diameter in l, l' respectively: MH, M'H meet the circle having BH as diameter in m, m' respectively; and NH, N'H meet the circle having CH as diameter in n, n' respectively. Prove that the following sets of points are concyclic:

HQLL'R, HRMM'P, HPNN'Q,
LL'MM'NN'll'mm'nn'.

J. A. THIRD.



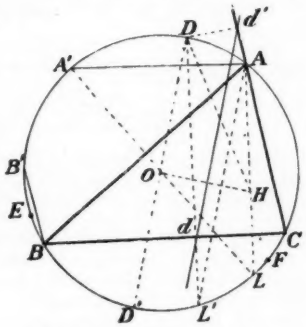
Solution by C. E. YOUNGMAN.

The incircle subtends a right angle at *L, L', M, M', N, N'*; ∴ these points lie on a circle, which the polar circle (being orthogonal to the director-circle of any in-conic) inverts into itself. The same inversion turns *BC* into the circle *AH*, and ∴ *L* into *l*, and so on; ∴ *ll'mm'nn'* are concyclic with *LL'MM'NN'*.

Let the tangents giving *Q* and *R* meet at *D*. They form, with *AB* and *AC*, a parallelogram circumscribing the incircle; ∴ the centre *I* is the mid-point of *AD*. But the perp. from *I* on *Ll* bisects it, because *IL* = *Il*; ∴ *DL* is, like *Al*, perp. to *Ll*. Similarly *DL'* to *L'l'*; ∴ *HD* subtends a right angle at *Q, R, L,* and *L'*; ∴ *HQLL'R* are concyclic.

339. [K. 2. c.] *D, E, F being the points whose pedal lines pass through the nine-point centre, show that in the circle ABC the arcs AD, BE, CF are one-third of the arcs AA', BB', CC' cut off by chords parallel to BC, CA, AB; and that the triangle DEF is equilateral.*

C. E. YOUNGMAN.



Solution by PROPOSER.

Let *H* and *O* be the ortho- and circum-centres. Draw the chords *AL, DL'* perp. to *BC*, and *DD'* parallel to *dd'* the pedal line of *D*. Then we have *AL'* parallel to *dd'* or *DD'*; hence arc *LL'* = *AD* = *L'D*. But *A'L* is a diameter, because $\angle LAA' = 90°$; and so is *DD'*, because *dd'* parallel to it bisects *HD* and *HO*:

∴ arc *DA'* = *LD'* = 2*AD*.
∴ 3*AD* = *AA'*.

By the same proof, 3*AE*, and 3*AF* also, = *AA'*; which must mean that

3*AE* = circumference + *AA'*,

and

3*AF* = twice circumference + *AA'*.

Hence *DE, EF, FD* each = $\frac{1}{3}$ circumference.

340. [K. 20. e.] *The Euler line of a triangle ABC meets BC at an angle such that* $\tan\alpha(\tan B - \tan C) = 3 - \tan B \tan C$. *Find the relation connecting the angles* α, β, γ *that the Euler line makes with the sides of the triangle.*

E. N. BARISIEN.

Solution by W. F. BEARD.

Put $a=\tan a, \ldots ; x=\tan A$, etc.

Then $a(y-z)=3-yz$, etc. ;

whence on reduction, $\Sigma ab=-3$ or $\Sigma \tan a \tan \beta+3=0$.

Solution by C. V. DURELL.

For B write $(\alpha-\gamma)$; for C, $(\beta-\alpha)$; then on reducing from

$$\tan \alpha\{\tan(\alpha-\gamma)-\tan(\beta-\alpha)\}=3-\tan(\alpha-\gamma)\tan(\beta-\alpha)$$

we get $\Sigma \cos(\beta+\gamma-\alpha)=0$.

342. [I. 18. a. c.] *The sum or difference of two perfect cubes is, or has a factor, of the form* $3m^2+n^2$. R. F. DAVIS

Solution by Proposer, C. V. DURELL.

Every factor of a number of the quadratic form $3m^2+n^2$ is of the same form.

Since $3(x+y)^2+(x-y)^2=4(x^2+xy+y^2)$, it follows that x^2+xy+y^2 is of the form $3m^2+n^2$.

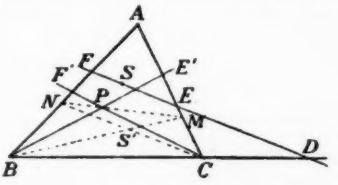
Now $x^3-y^3=x^2+xy+y^2$ when $x-y=1$; and contains x^2+xy+y^2 as a factor in other cases. Hence etc.

Similarly for x^3+y^3.

343. [L¹. 7. d.] *If one focus of an in-conic lie on the Euler line of the triangle, find the locus of the second focus.* J. M. DYER.

Solution by C. E. YOUNGMAN.

Let the Euler line (or any other) of ABC cut the sides at D, E, F ; and let S on DEF be one focus of an in-conic, and S' the other, so that BS, BS' are isogonal in the angle B, and CS, CS' in C ; draw also BE' isogonal to BE, and CF' to CF. Then, in cross-ratio, pencil $B(SDEF)=B(S'AE'C)$, being equiangular ; and



$$C(SDEF)=C(S'ABF') ;$$

$$\therefore \quad B(S'AE'C)=C(AS'F'B) ;$$

$\therefore$ M, N, P, the intersections of (BS' and CA), (BA and CS'), (BE' and CF'), are collinear. But P is fixed : hence an easy construction for any number of points S'.

Their locus is a conic ABC, as the sides of $S'MN$ pass through fixed points P, B, C, and two of its vertices lie on fixed lines CA, AB.

348. [K. 9. a.] *If a polygon be described having its sides equal, parallel, and in the same sense, but not in the same order, as the sides of a given closed polygon, the first-mentioned polygon is also closed.* F. S. MACAULAY.

Solution by J. F. HUDSON.

Let a_1, a_2, a_3, ... a_n be the sides of the given closed polygon, making α, β, γ, ... κ with any line in the plane of the polygon.

Then $\Sigma a_n \cos \kappa=0=\Sigma a_n \sin \kappa$

are the conditions that the polygon may be closed ; and they are independent of the *order* in which the sides are taken, if the direction and sense are unchanged.

This may be extended to three dimensions, since

$$\Sigma a_n \lambda_n=\Sigma a_n \mu_n=\Sigma a_n \nu_n=0,$$

where a_1, a_2, ... a_n are the lengths of lines whose direction cosines are

$$(\lambda_1, \mu_1, \nu_1) \ldots (\lambda_n, \mu_n, \nu_n).$$

351. [A. 1. b.] *If m be the geometric mean of n quantities $a, b, \ldots k$, prove that*

$$\Sigma\frac{(a-b)^2}{ab} \leqslant n\,\Sigma\frac{(a-m)^2}{am}$$

where the first summation extends to every pair of the quantities.

R. F. MUIRHEAD.

Solution by R. F. DAVIS.

Since the A.M. of any number of positive quantities $\leqslant$ G.M.,

$$\{\Sigma a - nm\}\{\Sigma(1/a) - n/m\} \leqslant 0.$$

But $$\Sigma a \,.\, \Sigma(1/a) = \Sigma(a/b) + n.$$

Hence $$\Sigma(a/b) - n(n-1) \leqslant n\{\Sigma(a).(1/m) + m\Sigma(1/a) - 2n\},$$

which is the required inequality.

352. [K. 1. c.] *Given three points A, B, C, and two parallel straight lines; find a point P on one of the lines, such that if PA, PB, PC cut the other line in α, β, γ, then shall $\alpha\beta = \beta\gamma$.* A. W. S.

Solution by R. F. DAVIS.

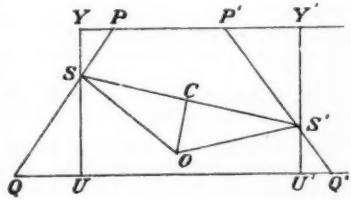
Let EF, GH be the given parallels, upon the first of which it is required that P should lie. Produce BC to meet EF in Q, and take R upon BC so that $\{RBCQ\} = -1$. Again, produce RA to meet EF in P, which will be the required point. Then $P\{\alpha\beta\gamma\infty\}$ estimated upon the transversal GH

$$= P\{RBCQ\} = -1\,; \text{ and } \alpha\beta = \beta\gamma.$$

353. [L¹. 17. e.] *Two parallel tangents to a conic meet a variable bitangent circle in four points. Show two of the connectors of these points pass through the foci of the conic, and that the remaining two are each equal to the focal axis of the conic, and are fixed in direction.* J. A. THIRD.

Solution by R. F. DAVIS, W. J. GREENSTREET, and others.

Let $YUU'Y'$ be the rectangle formed by two fixed parallel tangents YY', UU' to a given ellipse (centre C); and the perpendiculars YU, $Y'U'$ passing respectively through the foci S, S'. Then YCU', $Y'CU$ are diagonals of the auxiliary circle (centre C, radius CA). Through S, S' respectively draw PSQ, $P'SQ'$ making the angle $PSY = P'SY' = \theta$. Then by the theory of similar figures since Y, Y', U, U' lie on a circle centre C, radius CS/e, therefore P, P', Q, Q' lie on a circle centre O found by drawing CO perpendicular to SS' and making the angle $OSC = OS'C = \theta$, and radius SO/e or $S'O/e$. But this is the well-known construction for a bitangent circle (*Milne and Davis*, p. 162).



Conversely; etc., noting that $P'Q$ is equal and parallel to a fixed diameter $Y'U$ of the auxiliary circle.

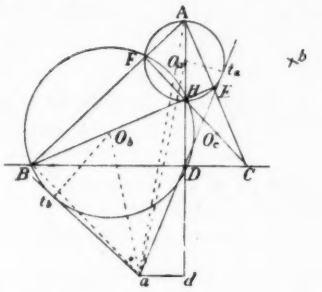
354. [K. 11. a.] *If the radii of two orthogonal circles with fixed centres vary, find the envelope of their common tangents.* A. THOMPSON.

Solution by R. F. DAVIS, Proposer, and others.

Evidently the sum of the squares of the perpendiculars from two fixed points (the centres of the given circles) upon any common tangent is constant, being equal to the square of the distance between the centres O, O'. The

envelope is therefore an ellipse having OO' for its minor axis. For if the sum of the squares of the perpendiculars from two opposite angular points of a square upon any variable line is constant, the product of the perpendiculars from the other angular points upon the same straight line is constant (*Casey's Sequel*, Book II., Prop. 8).

355. [**K. 20. e.**] *a, b, c are the centres of the squares described externally on the sides BC, CA, AB of a triangle ABC. H is the orthocentre of the triangle, and D, E, F the feet of the perpendiculars from A, B, C on BC, CA, AB. Tangents are drawn from a, b, c to the circles $HEAF$, $HFBD$, $HDCE$. If $t_r (r=1, 2, \dots 18)$ be one of the tangents, $\Sigma t_r^2 = 4\Delta(2\cot\omega + 3)$.*



R. TUCKER.

Solution by PROPOSER.

$$(at_b)^2 = (aO_b)^2 - \rho_b^2$$
$$= (aB)^2 + a\rho_b(\cos C - \sin C)$$
$$= 2R^2[\sin^2 A + \sin A \cos B \cos C - \sin A \cos B \sin C]. \quad \dots\dots\dots\dots\dots\dots\text{(i)}$$
$$(at_c)^2 = 2R^2[\sin^2 A + \sin A \cos B \cos C - \sin A \sin B \cos C]. \quad \dots\dots\dots\dots\dots\dots\text{(ii)}$$
$$(at_a)^2 = (aO_a)^2 - \rho_a^2$$
$$= 2R^2[\sin^2 A + \sin A \cos\{B - C\}].$$
$$(aO_a)^2 = (dO_a)^2 + (da)^2$$
$$= R^2[\{\cos(B - C) + \sin A\}^2 + \{\sin(B - C)\}^2]$$
$$= R^2[1 + \sin^2 A + 2\sin A \cos(B - C)].$$
$$\therefore\ (at_a)^2 + (at_b)^2 + (at_c)^2 = 2R^2[3\sin^2 A + 3\sin A \cos B \cos C - \sin^2 A + \Pi \sin A]$$
$$= 2R^2[2\sin^2 A + 3\sin A \cos B \cos C + \Pi \sin A].$$
$$\therefore\ \Sigma(t)^2 \text{ (18 tangents)}$$
$$= 8R^2\Sigma(\sin^2 A) + 24R^2\Pi \sin A$$
$$= 2\Sigma(a^2) + 12\Delta = 4\Delta\{2\cot\omega + 3\}.$$

357. [**K. 11. a; 20. e.**] *A, B, C are the centres of three circles, radii r_1, r_2, r_3, and the radius of their orthogonal circle is ρ. Shew that*

$$16\Delta^2\rho^2 = \Sigma a^2 r_1^4 - 2\Sigma r_2^2 r_3^2 bc \cos A - 2abc\Sigma r_1^2 a \cos A + a^2b^2c^2$$

where Δ, a, A etc. refer to the triangle ABC. A. C. L. WILKINSON.

Solution by R. F. DAVIS.

If p, q, r are the distances of any variable point from the angular points of a triangle ABC, they are connected by the relation

$$o = -a^2b^2c^2 + a^2p^2(b^2 + c^2 - a^2) + \dots + \dots - a^2(p^2 - q^2)(p^2 - r^2) \dots .$$

(Todhunter's *Spherical Trig.*)

This relation may be written more conveniently

$$a^2b^2c^2 + a^2p^4 + \dots + \dots = 2bc\cos A(q^2r^2 + a^2p^2) + \dots + \dots .$$

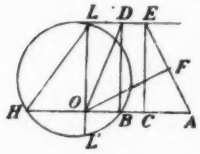
If now we put $p^2 = \rho^2 + r_1^2$, $q^2 = \rho^2 + r_2^2$, $r^2 = \rho^2 + r_3^2$ we get the equation in ρ as given in the question.

358. [K. 1. c.] *O, B, C, A are fixed collinear points; BD, CE are perpendiculars to OA. Find DE parallel to OA if OF is perpendicular to AE and equal to OD.*

R. B. WORTHINGTON.

Solution by R. F. DAVIS, PROPOSER, and many others.

Produce AO to H so that $OH=AC$; and through B, H draw a circle of diameter OA. Let LOL' be the chord of this circle perpendicular to OA. Through L draw LDE parallel to OA. Then



$$BL \cdot HL = OL \cdot OA; \text{ or}$$
$$BL : OA = OL : HL = CE : AE.$$

But $OF : OA = CE : AE$. Hence $OF = BL = OD$.

A second solution is obtained by drawing a parallel through L'.

359. [K. 20. c.] *If the roots of* $\tan\theta = m\tan(A+\theta) + n\tan(B+\theta)$ *may be said to be* α, β, γ, *then*

$$\sin(A-B) \cdot \Pi\sin\alpha = \sin A \Pi\sin(B+\alpha) - \sin B \Pi\sin(A+\alpha).$$

C. E. YOUNGMAN.

Solution by R. F. DAVIS.

Transform the given equation by means of the substitution $x = \cot\theta + \cot A$, and reduce. It will then be found that the resulting cubic in x gives

$$x_1x_2x_3 = (a^2+1) \cdot \{na(a-b)+ab+1\}/(na+mb),$$

where $a = \cot A$, $b = \cot B$.

But $$x_1x_2x_3 = \Pi\sin(A+\alpha)/\{\sin^3 A \cdot \Pi\sin\alpha\}.$$

Thus $$\Pi\sin(A+\alpha)/\{\sin A \cdot \Pi\sin\alpha\} \quad \text{.....(i)}$$
$$= \{na(a-b)+ab+1\}/(na+mb).$$

Similarly by putting $y = \cot\theta + \cot B$ it will be found that

$$\Pi\sin(B+\alpha)/\{\sin B \cdot \Pi\sin\alpha\} \quad \text{.....(ii)}$$
$$= \{mb(b-a)+ab+1\}/(na+mb).$$

Hence (ii) − (i) $= b - a$; etc.

361. [J. a. a.] *At the corners and the middle points of the sides of a square, side 400 yds., are placed 8 guns. An officer, starting from the centre of the square, inspects the guns successively, but never two guns consecutively in the same side of the square. Find the least distance he has to walk.* $[\sqrt{2} = \frac{7}{5}; \sqrt{5} = \frac{9}{4}.]$

R. F. DAVIS.

Solution by PROPOSER.

Let a, b, c, d be the mid-points in order of the sides of the square $ABCD$, centre O. There are many ways in which the officer can inspect the guns, *e.g.* $OACadcbDB$, $OAcBdCaDb$, $OadBDbcAC$, $OAcBDaCdb$, $OAcaCdbDB$, ... each step ranging in length from AC, through Ab, bd, to ab.

The route $OACadcbDB$ requires three such shortest steps, and will be the minimum. And it is $200\{\frac{7}{5}+\frac{14}{5}+\frac{9}{4}+\frac{21}{5}+\frac{9}{4}+\frac{14}{5}\}$ yds. $= 3140$ yds.

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